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Kronecker products of representations of exceptional Lie groups

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Abstract. A technique is described for evaluating Kronecker products of irreducible representations of an exceptional Lie group, G. The method depends upon the natural embedding in G of a classical Lie subgroup H. The problem is reduced to that of finding the branching of one irreducible representation of G on restriction to H, evaluating Kronecker products in H and finally using the modification rules appropriate to G.

1. Introduction

In two previous papers (King and Al-Qubanchi 1981a, b, hereafter referred to as I and II) some properties of the irreducible representations of the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 were discussed. However, the problem of evaluating Kronecker or tensor product multiplicities was not touched upon, even though its importance had been pointed out by Wybourne and Bowick (1977). An extensive tabulation of such products for G_2 had been given earlier by Butler (Wybourne 1970) and some similar results for the remaining exceptional groups were given by Wybourne and Bowick (1977) and extended considerably in the case of E_8 by Wybourne (1979). In the case of G_2 use was made of the embedding of this group in a classical group but in the other cases the technique employed involved the branching from an exceptional Lie group, G, to a maximally embedded classical Lie subgroup H, the evaluation of Kronecker products for H and the inversion of the branching from G to H.

An alternative procedure devised by Racah (1964) and Speiser (1964), applicable to all semi-simple Lie groups including the exceptional ones, is based directly on the formula due to Weyl (1926) for the characters of irreducible representations. Its implementation depends upon a knowledge of the Weyl symmetry group W_G of the Lie group G under consideration and the weight multiplicities of the irreducible representations of G. This procedure, described in the following section, has been used by Al-Qubanchi (1978) to obtain some preliminary results for the exceptional Lie groups and it forms the basis of the compilation of more extensive results in preparation by Englefield (1981).

Yet another procedure, which is in a sense intermediate between that of Wybourne and Bowick and that of Racah and Speiser, is described here in § 3 and illustrated in § 4. Some of its advantages are discussed in the concluding section.

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2. The Racah–Speiser method

The character of the irreducible representation $\boldsymbol{\lambda}_{\mathrm{G}}$ of the compact semi-simple Lie group, G, having highest weight $\boldsymbol{M}_{\mathrm{G}}$ is given, in the class specified by parameters $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_d)$, by the formula due to Weyl (1926):

$$\chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi}) = \frac{\Sigma_{S \in \mathrm{W}_{\mathrm{G}}} \eta_S \exp[\mathrm{i}S(\boldsymbol{R}_{\mathrm{G}} + \boldsymbol{M}_{\mathrm{G}}) \cdot \boldsymbol{\phi}]}{\Sigma_{S \in \mathrm{W}_{\mathrm{G}}} \eta_S \exp(\mathrm{i}S\boldsymbol{R}_{\mathrm{G}} \cdot \boldsymbol{\phi})} \tag{2.1}$$

where the summations are carried out over all elements S of the Weyl group W_{G}, $\eta_S = \pm 1$ is the parity of S and $\boldsymbol{R}_{\mathrm{G}}$ is half the sum of the positive roots of the corresponding Lie algebra. The expansion of this character, which serves to define each weight $\boldsymbol{m}$ and its multiplicity $M_{\boldsymbol{m}}^{\lambda_{\mathrm{G}}}$ in the representation $\boldsymbol{\lambda}_{\mathrm{G}}$, takes the form

$$\chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi}) = \sum_{\boldsymbol{m}} M_{\boldsymbol{m}}^{\lambda_{\mathrm{G}}} \exp(\mathrm{i}\boldsymbol{m} \cdot \boldsymbol{\phi}). \tag{2.2}$$

Using the fact that for all elements S of W_{G}

$$(S\boldsymbol{v}) \cdot (S\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{w} \tag{2.3}$$

for all $\boldsymbol{v}$ and $\boldsymbol{w}$, it is a straightforward task to derive from (2.1) the Weyl symmetry conditions

$$\chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi}) = \chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(S\boldsymbol{\phi}) \tag{2.4}$$

and

$$M_{\boldsymbol{m}}^{\lambda_{\mathrm{G}}} = M_{S\boldsymbol{m}}^{\lambda_{\mathrm{G}}}. \tag{2.5}$$

It follows that (2.2) can equally well be written in the form

$$\chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi}) = \sum_{\boldsymbol{m}} M_{\boldsymbol{m}}^{\lambda_{\mathrm{G}}} \exp(\mathrm{i}S\boldsymbol{m} \cdot \boldsymbol{\phi}). \tag{2.6}$$

Using (2.1) and (2.6) the Kronecker product of the irreducible representations $\boldsymbol{\lambda}_{\mathrm{G}}$ and $\boldsymbol{\mu}_{\mathrm{G}}$ may be analysed by writing

$$\chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi})\chi_{\mathrm{G}}^{\mu_{\mathrm{G}}}(\boldsymbol{\phi}) = \frac{\Sigma_{S \in \mathrm{W}_{\mathrm{G}}} (\eta_S \exp[\mathrm{i}S(\boldsymbol{R}_{\mathrm{G}} + \boldsymbol{M}_{\mathrm{G}}) \cdot \boldsymbol{\phi}] \, \Sigma_{\boldsymbol{m}} M_{\boldsymbol{m}}^{\mu_{\mathrm{G}}} \exp(\mathrm{i}S\boldsymbol{m} \cdot \boldsymbol{\phi}))}{\Sigma_{S \in \mathrm{W}_{\mathrm{G}}} \eta_S \exp(\mathrm{i}S\boldsymbol{R}_{\mathrm{G}} \cdot \boldsymbol{\phi})}. \tag{2.7}$$

This result gives the formula exploited by Speiser (1964) to evaluate Kronecker products, namely

$$\chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi})\chi_{\mathrm{G}}^{\mu_{\mathrm{G}}}(\boldsymbol{\phi}) = \sum_{\boldsymbol{m}} M_{\boldsymbol{m}}^{\mu_{\mathrm{G}}} \chi_{\mathrm{G}}^{\lambda_{\mathrm{G}} + P\boldsymbol{m}}(\boldsymbol{\phi}) \tag{2.8}$$

where the operator P merely projects the weight vector $\boldsymbol{m}$ from the root space V to the representation label space Λ as described in II.

The implementation of (2.8) to give the Kronecker product multiplicities $K_{\nu_{\mathrm{G}}}^{\lambda_{\mathrm{G}}\mu_{\mathrm{G}}}(\mathrm{G})$ of G defined by

$$\chi_{\mathrm{G}}^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi})\chi_{\mathrm{G}}^{\mu_{\mathrm{G}}}(\boldsymbol{\phi}) = \sum_{\nu_{\mathrm{G}}} K_{\nu_{\mathrm{G}}}^{\lambda_{\mathrm{G}}\mu_{\mathrm{G}}}(\mathrm{G}) \chi_{\mathrm{G}}^{\nu_{\mathrm{G}}}(\boldsymbol{\phi}) \tag{2.9}$$

merely requires a knowledge of the weight multiplicities $M_{\boldsymbol{m}}^{\mu_{\mathrm{G}}}$ and the modification rules defined by the identity

$$\chi_{\mathrm{G}}^{\lambda}(\boldsymbol{\phi}) = \eta_S \chi_{\mathrm{G}}^{PSP^{-1}(\boldsymbol{\lambda} + \boldsymbol{\delta}_{\mathrm{G}}) - \boldsymbol{\delta}_{\mathrm{G}}}(\boldsymbol{\phi}) \tag{2.10}$$

for each element S of W_G. The notation here is such that $\boldsymbol{\lambda}$ is any vector in the space Λ and $\boldsymbol{\delta}_G$ is the projection of $\boldsymbol{R}_G$ into this space, just as $\boldsymbol{\lambda}_G$ is the projection of $\boldsymbol{M}_G$; that is,

$$\boldsymbol{\delta}_G = P\boldsymbol{R}_G \qquad \text{and} \qquad \boldsymbol{\lambda}_G = P\boldsymbol{M}_G. \tag{2.11}$$

For each vector $\boldsymbol{\lambda}$ there exists some S such that in (2.10) either

$$\chi_G^{\lambda}(\boldsymbol{\phi}) = \eta_S \chi_G^{\lambda}(\boldsymbol{\phi}) = -\chi_G^{\lambda}(\boldsymbol{\phi}) = 0 \tag{2.12}$$

or

$$\chi_G^{\lambda}(\boldsymbol{\phi}) = \eta_S \chi_G^{\nu_G}(\boldsymbol{\phi}) \tag{2.13}$$

with $\boldsymbol{\nu}_G$ a unique G-dominant vector as defined in I. This property implies, on comparing (2.8) and (2.9), and making use of (2.10) with $\boldsymbol{\lambda} = \boldsymbol{\lambda}_G + P\boldsymbol{m}$, the validity of the formula due to Racah (1964):

$$K_{\nu_G}^{\lambda_G \mu_G}(G) = \sum_{S \in W_G} \eta_S M_{SP^{-1}(\nu_G+\delta_G)-P^{-1}(\lambda_G+\delta_G)}^{\mu_G}. \tag{2.14}$$

3. Branching rule method

In order to derive for each exceptional simple Lie group G a method of calculating Kronecker product multiplicities which does not depend on a complete knowledge of weight multiplicities and the subsequent use of a very large number of Weyl symmetry operations, it is convenient to make use of the properties of a naturally embedded classical semi-simple Lie subgroup H. Such embeddings were defined and discussed in detail in I and II. The corresponding class parameters of G and H coincide, so that the branching multiplicities, $B_{\sigma_H}^{\mu_G}$, are defined through the identity

$$\chi_G^{\mu_G}(\boldsymbol{\phi}) = \sum_{\sigma_H} B_{\sigma_H}^{\mu_G} \chi_H^{\sigma_H}(\boldsymbol{\phi}) \tag{3.1}$$

where the summation is carried out over irreducible representations $\boldsymbol{\sigma}_H$ of H. In addition, the Weyl group W_G of G contains the Weyl group W_H of H as a subgroup so that

$$W_G = \bigcup_{\gamma=1}^{c} T_\gamma W_H$$

where T_γ, for $\gamma = 1, 2, \ldots, c = |W_G|/|W_H|$, are left coset representatives.

Using the coset decomposition the numerator of the character (2.1) may be written in the form

$$\begin{aligned}
\xi_G^{M_G}(\boldsymbol{\phi}) &= \sum_{\gamma=1}^{c} \sum_{T \in W_H} \eta_{T_\gamma} \eta_T \exp[\mathrm{i} T_\gamma T(\boldsymbol{R}_G + \boldsymbol{M}_G) \cdot \boldsymbol{\phi}] \\
&= \sum_{\gamma=1}^{c} \sum_{T \in W_H} \eta_{T_\gamma} \eta_T \exp[\mathrm{i} T(\boldsymbol{R}_G + \boldsymbol{M}_G) \cdot T_\gamma^{-1} \boldsymbol{\phi}] \\
&= \sum_{\gamma=1}^{c} \sum_{T \in W_H} \eta_{T_\gamma} \eta_T \exp[\mathrm{i} T(\boldsymbol{R}_H + \boldsymbol{R}_G - \boldsymbol{R}_H + \boldsymbol{M}_G) \cdot T_\gamma^{-1} \boldsymbol{\phi}] \\
&= \sum_{\gamma=1}^{c} \eta_{T_\gamma} \xi_H^{M_G + R_G - R_H}(T_\gamma^{-1} \boldsymbol{\phi})
\end{aligned} \tag{3.2}$$

where use has been made of (2.3), and $\boldsymbol{M}_{\mathrm{G}}+\boldsymbol{R}_{\mathrm{G}}-\boldsymbol{R}_{\mathrm{H}}$ is necessarily the highest weight of an irreducible representation $\boldsymbol{\tau}_{\mathrm{H}}$ of H, with

$$\boldsymbol{\tau}_{\mathrm{H}}=P(\boldsymbol{M}_{\mathrm{G}}+\boldsymbol{R}_{\mathrm{G}}-\boldsymbol{R}_{\mathrm{H}})=\boldsymbol{\lambda}_{\mathrm{G}}+\boldsymbol{\delta}_{\mathrm{G}}-\boldsymbol{\delta}_{\mathrm{H}}, \tag{3.3}$$

where use has been made of (2.11) and the fact that, for the natural embeddings of H in G used here, the projection operators P of G and H coincide. Thus, introducing

$$\xi^0_{\mathrm{G}}(\boldsymbol{\phi})=\sum_{S\in \mathrm{W_G}}\eta_S \exp(\mathrm{i}S\boldsymbol{R}_{\mathrm{G}}\cdot\boldsymbol{\phi})$$

and

$$\xi^0_{\mathrm{H}}(\boldsymbol{\phi})=\sum_{T\in \mathrm{W_H}}\eta_T \exp(\mathrm{i}T\boldsymbol{R}_{\mathrm{H}}\cdot\boldsymbol{\phi})$$

for the denominators of the characters of irreducible representations of G and H,

$$\chi^{\lambda_{\mathrm{G}}}_{\mathrm{G}}(\boldsymbol{\phi})=\sum_{\gamma=1}^{\mathrm{c}}\eta_{T_\gamma}\chi^{\tau_{\mathrm{H}}}_{\mathrm{H}}(T_\gamma^{-1}\boldsymbol{\phi})\xi^0_{\mathrm{H}}(T_\gamma^{-1}\boldsymbol{\phi})/\xi^0_{\mathrm{G}}(\boldsymbol{\phi}). \tag{3.4}$$

However, since T^{-1} is an element of $\mathrm{W_G}$, (2.4) and (3.1) imply that

$$\chi^{\mu_{\mathrm{G}}}_{\mathrm{G}}(\boldsymbol{\phi})=\chi^{\mu_{\mathrm{G}}}_{\mathrm{G}}(T_\gamma^{-1}\boldsymbol{\phi})=\sum_{\sigma_{\mathrm{H}}}B^{\mu_{\mathrm{G}}}_{\sigma_{\mathrm{H}}}\chi^{\sigma_{\mathrm{H}}}_{\mathrm{H}}(T_\gamma^{-1}\boldsymbol{\phi}). \tag{3.5}$$

It follows that

$$\chi^{\lambda_{\mathrm{G}}}_{\mathrm{G}}(\boldsymbol{\phi})\chi^{\mu_{\mathrm{G}}}_{\mathrm{G}}(\boldsymbol{\phi})=\sum_{\gamma=1}^{\mathrm{c}}\eta_{T_\gamma}\sum_{\sigma_{\mathrm{H}}}B^{\mu_{\mathrm{G}}}_{\sigma_{\mathrm{H}}}\sum_{\rho_{\mathrm{H}}}K^{\sigma_{\mathrm{H}}\tau_{\mathrm{H}}}_{\rho_{\mathrm{H}}}(\mathrm{H})\chi^{\rho_{\mathrm{H}}}_{\mathrm{H}}(T_\gamma^{-1}\boldsymbol{\phi})\xi^0_{\mathrm{H}}(T_\gamma^{-1}\boldsymbol{\phi})/\xi^0_{\mathrm{G}}(\boldsymbol{\phi}). \tag{3.6}$$

Comparison with (3.3) and (3.4) then gives precisely the result sought, namely an analogue of Speiser's formula (2.8):

$$\chi^{\lambda_{\mathrm{G}}}_{\mathrm{G}}(\boldsymbol{\phi})\chi^{\mu_{\mathrm{G}}}_{\mathrm{G}}(\boldsymbol{\phi})=\sum_{\sigma_{\mathrm{H}},\rho_{\mathrm{H}}}B^{\mu_{\mathrm{G}}}_{\sigma_{\mathrm{H}}}K^{\sigma_{\mathrm{H}},\lambda_{\mathrm{G}}+\delta_{\mathrm{G}}-\delta_{\mathrm{H}}}_{\rho_{\mathrm{H}}}(\mathrm{H})\chi^{\rho_{\mathrm{H}}-\delta_{\mathrm{G}}+\delta_{\mathrm{H}}}_{\mathrm{G}}(\boldsymbol{\phi}). \tag{3.7}$$

This formula requires for its implementation the branching multiplicities associated with just one irreducible representation $\boldsymbol{\mu}_{\mathrm{G}}$ of G, the evaluation of Kronecker products in the classical subgroup H and the application of the modification rules following from (2.10) in the case $\boldsymbol{\lambda}=\boldsymbol{\rho}_{\mathrm{H}}-\boldsymbol{\delta}_{\mathrm{G}}+\boldsymbol{\delta}_{\mathrm{H}}$. These allow each non-vanishing character on the right-hand side of (3.7) to be expressed in the form $\eta_S\chi^{\nu_{\mathrm{G}}}(\boldsymbol{\phi})$ with $\boldsymbol{\nu}_{\mathrm{G}}$ once again a unique irreducible representation label as required in (2.9).

The analogue of Racah's formula (2.14) is then

$$K^{\lambda_{\mathrm{G}}\mu_{\mathrm{G}}}_{\nu_{\mathrm{G}}}(\mathrm{G})=\sum_{\sigma_{\mathrm{H}}}B^{\mu_{\mathrm{G}}}_{\sigma_{\mathrm{H}}}\sum_{S}\eta_S K^{\sigma_{\mathrm{H}},\lambda_{\mathrm{G}}+\delta_{\mathrm{G}}-\delta_{\mathrm{H}}}_{PSP^{-1}(\nu_{\mathrm{G}}+\delta_{\mathrm{G}})-\delta_{\mathrm{H}}}(\mathrm{H}). \tag{3.8}$$

The range of values of S which give rise to non-vanishing terms in the second summation is much more restricted than in (2.14), since it is now required that $SP^{-1}(\boldsymbol{\nu}_{\mathrm{G}}+\boldsymbol{\delta}_{\mathrm{G}})-P^{-1}\boldsymbol{\delta}_{\mathrm{H}}$ be H-dominant, i.e. the highest weight of some irreducible representation of H. This will only be true if $SP^{-1}(\boldsymbol{\nu}_{\mathrm{G}}+\boldsymbol{\delta}_{\mathrm{G}})$ is itself H-dominant, and this in turn will only be true by virtue of the G-dominance of both $\boldsymbol{\nu}_{\mathrm{G}}$ and $\boldsymbol{\delta}_{\mathrm{G}}$, if S is one of the right coset representations S_γ with $\gamma=1, 2, \ldots, \mathrm{c}$ defined explicitly in I. Indeed, these coset representatives of $\mathrm{W_G}$ with respect to $\mathrm{W_H}$ were selected precisely by requiring that $S_\gamma\boldsymbol{v}_{\mathrm{G}}$ be H-dominant for all G-dominant vectors $\boldsymbol{v}_{\mathrm{G}}$. Hence

$$K^{\lambda_{\mathrm{G}}\mu_{\mathrm{G}}}_{\nu_{\mathrm{G}}}(\mathrm{G})=\sum_{\sigma_{\mathrm{H}}}B^{\mu_{\mathrm{G}}}_{\sigma_{\mathrm{H}}}\sum_{\gamma=1}^{\mathrm{c}}\eta_{S_\gamma}K^{\sigma_{\mathrm{H}},\lambda_{\mathrm{G}}+\delta_{\mathrm{G}}-\delta_{\mathrm{H}}}_{PS_\gamma P^{-1}(\nu_{\mathrm{G}}+\delta_{\mathrm{G}})-\delta_{\mathrm{H}}}(\mathrm{H}). \tag{3.9}$$

4. Illustration

An instructive example of the use of the technique provided in the previous section is provided by the problem of reducing the Kronecker product of the irreducible representations (2110) and $(\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ of F_4. The relevant classical subgroup is SO(9) and the branching rule for the representation $(\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ is

$$F_4 \to SO(9) \quad (\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}) \to [\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}]+[1110]+[1100]+[1000]+[\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}], \tag{4.1}$$

as given by Wybourne and Bowick (1977).

From the labelling scheme of roots defined in I, $\boldsymbol{\delta}_{F_4} = (\frac{11}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2})$ and $\boldsymbol{\delta}_{SO(9)} = (\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2})$, so that $\boldsymbol{\delta}_{F_4} - \boldsymbol{\delta}_{SO(9)} = (2000)$ and $\boldsymbol{\tau}_{SO(9)} = [4110]$. The next step in the exploitation of (3.7) therefore involves the evaluation of the Kronecker products of irreducible representations of SO(9):

$$[4110]\times\{[\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}]+[1110]+[1100]+[1000]+[\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}]\}.$$

This may be effected using techniques described elsewhere (Butler and Wybourne 1969, Wybourne 1970, King 1971, 1975) to give the following terms $\boldsymbol{\rho}_{SO(9)}$:

$$\begin{aligned}
&[\tfrac{11}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2}]+[\tfrac{11}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}]+[\tfrac{11}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}]\\
&\quad+[5220]+[5211]+[5210]+2[5111]+[5200]+2[5110]+[5100]\\
&\quad+[5000]+[\tfrac{9}{2}\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}]+[\tfrac{9}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}]+[\tfrac{9}{2}\tfrac{5}{2}\tfrac{1}{2}\tfrac{1}{2}]+3[\tfrac{9}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2}]+4[\tfrac{9}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}]+3[\tfrac{9}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}]\\
&\quad+[4221]+[4220]+2[4211]+3[4210]+4[4111]+[4200]+3[4110]\\
&\quad+3[4100]+[4000]+[\tfrac{7}{2}\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}]+[\tfrac{7}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}]+[\tfrac{7}{2}\tfrac{5}{2}\tfrac{1}{2}\tfrac{1}{2}]+3[\tfrac{7}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2}]+4[\tfrac{7}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}]\\
&\quad+3[\tfrac{7}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}]+[3220]+[3211]+[3210]+2[3111]+[3300]+2[3110]\\
&\quad+[3100]+[3000]+[\tfrac{5}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2}]+[\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}]+[\tfrac{5}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}].
\end{aligned} \tag{4.2}$$

The final step is the subtraction of $\boldsymbol{\delta}_{F_4} - \boldsymbol{\delta}_{SO(9)} = (2000)$ from each of these terms, $\boldsymbol{\rho}_{SO(9)}$, and the use of the modification rules (2.10) for F_4 given in I:

$$\begin{aligned}
(\nu_1\nu_2\nu_3\nu_4) &= -(\nu_1,\ \nu_3-1,\ \nu_2+1,\ \nu_4)\\
&= -(\nu_1,\ \nu_2,\ \nu_4-1,\ \nu_3+1)\\
&= -(\nu_1,\ \nu_2,\ \nu_3,\ -\nu_4-1)\\
&= -(\nu_1-\epsilon,\ \nu_2+\epsilon,\ \nu_3+\epsilon,\ \nu_4+\epsilon)
\end{aligned} \tag{4.3}$$

with

$$\epsilon = \tfrac{1}{2}(\nu_1-\nu_2-\nu_3-\nu_4+1).$$

In fact, it is only necessary to use the last of these rules in the first instance. The penultimate rule then serves to complete the necessary modifications and yields the required F_4 Kronecker product:

$$\begin{aligned}
(2110)\times(\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}) &= (\tfrac{7}{2}\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2})+(\tfrac{7}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2})+(\tfrac{7}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})\\
&\quad+(3200)+(3111)+(3110)+(3100)+(3000)+2(\tfrac{5}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2})+2(\tfrac{5}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})\\
&\quad+(2110)+2(2100)+(2000)+(\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})+(1000).
\end{aligned} \tag{4.4}$$

The origin of the multiplicities appearing in this expansion is exemplified by the evaluation of the coefficient of (2110) using (3.9). In this case $\boldsymbol{\nu}_{F_4} + \boldsymbol{\delta}_{F_4} = (\frac{15}{2}\frac{7}{2}\frac{5}{2}\frac{1}{2})$ and the application of $PS_\gamma P^{-1}$, as given in table 2 of II, for the three coset representatives labelled by $\gamma = 1$, 2 and 3 yields $(\frac{15}{2}\frac{7}{2}\frac{5}{2}\frac{1}{2})$, (7431) and $(\frac{13}{2}\frac{9}{2}\frac{7}{2}\frac{1}{2})$. The corresponding irreducible representations $\boldsymbol{\rho}_{\mathrm{SO}(9)}$, together with the parity factors η_{S_γ} are $+[4110]$, $-[\frac{7}{2}\frac{3}{2}\frac{3}{2}\frac{1}{2}]$ and $+[3220]$. The coefficients of these terms in (4.2) are 3, 3 and 1 respectively, confirming that

$$K_{(2110)}^{(2110)\ (\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})}(\mathrm{F}_4) = 3 - 3 + 1 = 1,$$

as indicated in (4.4).

5. Conclusion

One particular aspect of the labelling schemes described in I makes itself felt in carrying out the calculations of the previous section. For all of the schemes of I associated with singly-supplemented Dynkin diagrams, the lexicographic ordering used is such that the only non-vanishing component of $\boldsymbol{\delta}_{\mathrm{G}} - \boldsymbol{\delta}_{\mathrm{H}}$ is the first, so that $\boldsymbol{\delta}_{\mathrm{G}} - \boldsymbol{\delta}_{\mathrm{H}} = (\delta, 0, 0, \ldots, 0)$ with $\delta > 0$. This implies that after adding this vector to $\boldsymbol{\lambda}_{\mathrm{G}}$ to give $\boldsymbol{\tau}_{\mathrm{H}}$, calculating the Kronecker product with $\boldsymbol{\sigma}_{\mathrm{H}}$ as required in (3.7), and then subtracting the same vector to give $\boldsymbol{\rho}_{\mathrm{H}} - \boldsymbol{\delta}_{\mathrm{G}} + \boldsymbol{\delta}_{\mathrm{H}}$, the only vectors which are not G-dominant are those whose first component is too small. In these cases it is then necessary to reflect in the hyperplane perpendicular to the single simple root of G which is not a simple root of H. Such reflections are given explicitly in table 13 of I and are expressed, as in (4.3), in terms of the addition or subtraction of a parameter ϵ. No such modification is needed if $\epsilon > 0$, whilst if $\epsilon = 0$ the corresponding character of G vanishes identically.

If $\boldsymbol{\mu}_{\mathrm{H}}$ itself has a small first component then these modifications are particularly easy to incorporate in the analysis. They lead, for example, to the following general results:

$$\mathrm{G}_2 \quad (\boldsymbol{\lambda}) \times (1) = (\{\boldsymbol{\lambda}\} \,.\, \{1\}) + (\{\boldsymbol{\lambda}\} \,.\, \{1^2\}) + (\{\boldsymbol{\lambda}\}) - \alpha(\boldsymbol{\lambda})$$

where $\alpha = 1$ if $\lambda_1 = 2\lambda_2$ and 0 if $\lambda_2 \neq 2\lambda_2$;

$$\mathrm{E}_6 \quad (\lambda : \boldsymbol{\mu}) \times (1:1) = (\lambda + 1 : \{\boldsymbol{\mu}\} \,.\, \{1\}) + (\lambda : \{\boldsymbol{\mu}\} \,.\, \{1^4\}) + (\lambda - 1 : \{\boldsymbol{\mu}\} \,.\, \{1\});$$

$$\mathrm{E}_7 \quad (\boldsymbol{\lambda}) \times (1^2) = (\{\boldsymbol{\lambda}\} \,.\, \{1^2\}) + (\{\boldsymbol{\lambda}\} \,.\, \{1^6\})$$

$$(\boldsymbol{\lambda}) \times (21^6) = (\{\boldsymbol{\lambda}\} \,.\, \{21^6\}) + (\{\boldsymbol{\lambda}\} \,.\, \{1^4\}) - \beta(\boldsymbol{\lambda})$$

where $\beta = 1$ if $\lambda_1 > \lambda_2$ and $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7$ and 0 otherwise;

$$\mathrm{E}_8 \quad (\boldsymbol{\lambda}) \times (21^7) = (\{\boldsymbol{\lambda}\} \,.\, \{21^7\}) + (\{\boldsymbol{\lambda}\} \,.\, \{1^3\}) + (\{\boldsymbol{\lambda}\} \,.\, \{1^6\}) - \gamma(\boldsymbol{\lambda})$$

where $\gamma = 1$ if $\lambda_1 > \lambda_2$ and $\lambda_1 = 2\lambda_2 + 2\lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8$ and 0 otherwise, where the notation is such that $(\{\boldsymbol{\lambda}\} \,.\, \{\boldsymbol{\sigma}\})$ signifies firstly the evaluation of the product $\{\boldsymbol{\lambda}\} \,.\, \{\boldsymbol{\sigma}\}$ in accordance with the rules appropriate to the group $\mathrm{SU}(n)$ for the value of n determined by the nature of the classical subgroup H of G, and secondly the interpretation of the resulting terms $\boldsymbol{\rho}$ as characters $(\boldsymbol{\rho})$ of G provided that $\boldsymbol{\rho}$ is G-dominant, and discarding these terms completely if $\boldsymbol{\rho}$ is not G-dominant. In the special case of E_6 the character $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6) = (\lambda\mu_1\mu_2\mu_3\mu_4\mu_5)$ has been denoted by $(\lambda : \boldsymbol{\mu})$ for convenience.

These particular results therefore have the merit of reducing the problem of the evaluation of products in G to the evaluation of products in $\mathrm{SU}(n)$. That this has been achieved is due to the very simple nature of the branching rule from G to H for the particular irreducible representations $\boldsymbol{\mu}_{\mathrm{G}}$ of G considered here, together with the rather trivial operation of the modification rules in these cases. This degree of simplicity is not always encountered. However, these two features are symptomatic of the power of the method developed here in comparison with that used by Wybourne and Bowick and that used by Racah and Speiser. To be more precise, firstly the technique involves the knowledge of the branching from G to H of just one representation $\boldsymbol{\mu}_G$, and no inversion of this branching rule is required. This contrasts with the method of Wybourne and Bowick, which requires a knowledge of the branching from G to H of both $\boldsymbol{\lambda}_G$ and $\boldsymbol{\mu}_G$ together with the inversion of this branching for a large number of representations $\boldsymbol{\rho}_H$ of H. Secondly, the technique involves the use of modification rules determined essentially by a single Weyl reflection, and at most by no more reflections than the index, c, of W_H in W_G. This contrasts with the multitude of Weyl reflections which need to be used in conjunction with the method of Racah and Speiser, which also requires a knowledge of weight multiplicities. Finally, it should be pointed out that for each exceptional group G the possible maximally embedded classical subgroups H have been established in I. The simplest choice of H is SU(3), SO(9), SU(2) × SU(6), SU(8) and SO(16) in the cases for which G is G_2, F_4, E_6, E_7 and E_8 respectively, as explained in II. However, in the case of E_8 the subgroup SU(9) may be preferred to SO(16) by virtue of the ease with which Kronecker products may be evaluated for SU(9) representations using *S*-function techniques (Wybourne 1970).

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